

On the cycles of algebraic surfaces;

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Fourth supplement to the *Analysis Situs*.

§ 1. — Introduction.

The fine works of Mr. Picard on *Algebraic surfaces* have long since brought to light the importance of the notion of cycles of one, two or three dimensions. I thought that one might apply to this question the principles that I set out in the *Analysis Situs* and its first two supplements (*Journal de l'École Polytechnique*, Centenary volume; *Rendiconti del Circolo matematico di Palermo*, vol. XIII; *Proceedings of the London mathematical Society*, Vol. XXXII), and I have thus obtained certain partial results, which I have already stated in a Note in the *Comptes rendus* and which come to complete, on certain points, those of Mr. Picard.

I recall that, a closed manifold V of p dimensions being given, I draw on this manifold other manifolds, closed or not, of a smaller number of dimensions; I denote by W_q a manifold of q dimensions drawn in this way on V .

If ΣW_q is a set of manifolds of q dimensions and ΣW_{q-1} a set of manifolds of $q - 1$ dimensions, the congruence

$$\Sigma W_q \equiv \Sigma W_{q-1}$$

signifies (by definition) that ΣW_{q-1} forms the complete boundary of the set of manifolds ΣW_q . I express the same fact, without bringing ΣW_q into evidence, by writing the relation

$$\Sigma W_{q-1} \sim 0$$

which I call *a homology*.

Then the congruence

$$\Sigma W_q \equiv 0$$

will signify that the manifold ΣW_q is closed.

If one has $\Sigma W_q \equiv 0$ without having $\Sigma W_q \sim 0$ (or $n\Sigma W_q \sim 0$, n being an integer), I shall say that the manifold ΣW_q constitutes a cycle of q dimensions.

Let

$$f(x, y, z) = 0 \tag{1}$$

be the equation of any algebraic surface, which will define a manifold V of four dimensions. To each value of y there will correspond a Riemann surface, which will in general be of genus p . I shall suppose that the genus is lowered neither for $y = 0$ nor for $y = \infty$, but that it is lowered for q singular points.

$$y = A_1, \quad y = A_2, \quad \dots, \quad y = A_q.$$

In the y -plane I shall draw q cuts $OA_1, OA_2, \dots, OA_q$. Let S be one of the Riemann surfaces; when y varies *without crossing the cuts*, the surface S will

vary, but while remaining homeomorphic to itself, in such a way that the various Riemann surfaces correspond to one another point by point, in a one-to-one and continuous manner.

Any one S of these surfaces can be subdivided into a polyhedron P ; let F be the faces, B the edges, C the vertices of this

polyhedron. Another surface S' , corresponding point by point to the surface S , will likewise be found subdivided into a polyhedron P' , whose faces, edges and vertices will correspond to the faces, the edges and the vertices of the polyhedron P .

Let us now suppose that y , starting from a point situated infinitely near one of the cuts, describes an almost closed contour and ends at another point situated infinitely near the initial point, *but on the other side of the cut*. The surface S will have been transformed into an infinitely little different surface; but to a point of the first surface there will correspond, in general, a point of the second surface which will differ from it greatly.

The polyhedron P will thus have been transformed into a very different polyhedron P' .

On the other hand, to the different points of the y -plane cut up by our cuts, we can make correspond the points of a polygon Q of $2q$ sides $\alpha_i\beta_i$ and $\alpha_i\beta_{i+1}$, the sides $\alpha_i\beta_i$ and $\alpha_i\beta_{i+1}$ corresponding to the two lips of the cut OA_i , the point α_i to A_i , the points β_i and β_{i+1} to O . Needless to add that I shall write indifferently β_1 or β_{q+1} , β_2 or β_{q+2} , etc., so as to preserve the symmetry of the notations.

From this we are going to derive a subdivision of the manifold V into a polyhedron H of four dimensions.

To each face F_i of P there will correspond a hypercell of R^1 which I shall also call F_i . To each edge B_i of P there will correspond a cell B_i of H ; likewise, to each of the sides $\alpha_i\beta_i$ or $\alpha_i\beta_{i+1}$ of Q , combined with each of the faces F_k of P , there will correspond a cell $\alpha_i\beta_iF_k$ or $\alpha_i\beta_{i+1}F_k$ of H .

To each vertex C_i of P there corresponds a face C_i of H . To each vertex α_i or β_i of Q , combined with each of the faces F_k of P , there will correspond a face α_iF_k or β_iF_k of H . To each side $\alpha_i\beta_i$ or $\alpha_i\beta_{i+1}$ of Q , combined with each of the edges B_k of P , there will correspond a face $\alpha_i\beta_i$ or $\alpha_i\beta_{i+1}B_k$ of H .

To each vertex C_k of P , combined with each of the sides $\alpha_i\beta_i$ or $\alpha_i\beta_{i+1}$ of H^2 , there will correspond an edge $\alpha_i\beta_iC_k$ or $\alpha_i\beta_{i+1}C_k$ of H . To each vertex α_i or β_i of Q , combined with each of the edges B_k of P , there will correspond an edge $\alpha_i\beta_k^3$ or β_iB_k of H .

Finally, to each vertex α_i or β_i of Q , combined with each of the vertices C_k of P , there will correspond a vertex α_iC_k or β_iC_k of H .

But several observations are in order. First, for $y = A_i$ the polyhedron P

¹*Editorial note:* Printed R ; the polyhedron being built is H , so H is expected; an apparent misprint.

²*Editorial note:* Printed H ; the sides $\alpha_i\beta_i$ and $\alpha_i\beta_{i+1}$ belong to the polygon Q , so Q is expected; an apparent misprint.

³*Editorial note:* Printed $\alpha_i\beta_k$; by the pattern of the sentence and of the Tableau on p. 173, α_iB_k is expected; an apparent misprint.

degenerates in such a way that some of its faces disappear; if, for example, the face F_k disappears for $y = A_i$, the corresponding face $\alpha_i F_k$ of the polyhedron H will not exist.

Likewise, although this can be avoided, one could conceive that an edge B_k should disappear for $y = A_i$; in that case, the edge $\alpha_i B_k$ would not exist.

On the other hand, let us give i a determinate value and make the index k take all possible values; let us then consider, on the one hand, the set of the cells $\alpha_i \beta_i F_k$ and, on the other hand, the set of the cells $\alpha_i \beta_{i+1} F_k$.

These two sets will be identical, although in general the cell $\alpha_i \beta_i F_k$, considered separately, is not identical to the cell $\alpha_i \beta_{i+1} F_k$, nor even to another cell $\alpha_i \beta_{i+1} F_l$.

It may also happen that some of the faces $\alpha_i \beta_i B$ are identical to some of the faces $\alpha_i \beta_{i+1} B$, or some of the edges $\alpha_i \beta_i C$ to some of the edges $\alpha_i \beta_{i+1} C$.

On the other hand, let us compare the different faces $\beta_i F_k$; the Riemann surface S_0 which corresponds to the point O will be found subdivided into a polyhedron in q different ways, according as one regards the point O as corresponding to the vertex β_1 , or to β_2 , ..., or to β_q . It is these q modes of subdivision which generate the faces βF . If, then, m is the number of the faces of P , one will have the identities

$$\beta_i F_1 + \beta_i F_2 + \dots + \beta_i F_m = \beta_j F_1 + \beta_j F_2 + \dots + \beta_j F_m \quad (2)$$

$(i, j = 1, 2, \dots, q).$

It may happen that some of the faces βF are identical to one another. But this will not always happen. It may likewise happen that some of the edges βB , or some of the vertices βC , are identical.

Finally, in consequence of the degeneration of P for $y = A_i$, it may happen that some of the faces $\alpha_i F$, or of the edges $\alpha_i B$, or of the vertices $\alpha_i C$ are identical.

In summary, our partial manifolds, hypercells, cells, edges or vertices can be distributed into four categories in accordance with the following Table:

Nature of the manifold.	Category			
	1.	$\alpha\beta.$	$\alpha.$	$\beta.$
Hypercells...	F_k	$\dot{z}$	$\dot{z}$	$\dot{z}$
Cells...	B_k	$\alpha_i \beta_i F_k, \alpha_i \beta_{i+1} F_k$	$\dot{z}$	$\dot{z}$
Faces...	C_k	$\alpha_i \beta_i B_k, \alpha_i \beta_{i+1} B_k$	$\alpha_i F_k$	$\beta_i F_k$
Edges...	$\dot{z}$	$\alpha_i \beta_i C_k, \alpha_i \beta_{i+1} C_k$	$\alpha_i B_k$	$\beta_i B_k$
Vertices...	$\dot{z}$	$\dot{z}$	$\alpha_i C_k$	$\beta_i C_k$

There can be no identity between two manifolds of different category. Two manifolds of category 1 are always distinct.

Between two manifolds of category $\alpha\beta$ there can be identity only if the index i of α is the same (otherwise the corresponding values of y would lie

on two different cuts OA_i, OA_j); the index of β must, on the contrary, be different; there can be identity, for example, between $\alpha_i\beta_i F_k$ and $\alpha_i\beta_{i+1} F_h$, but not between $\alpha_i\beta_i F_k$ and $\alpha_i\beta_i F_h$. Two manifolds of category α can be identical only if the index of α is the same.

Before going further, we are going to modify our conventions a little, in order to avoid the inconveniences which might result from identities such as (2), which hold between two sums of faces although the faces taken individually are not identical two by two.

Let M be any point of the cut OA_i ; the corresponding Riemann surface S can be decomposed into a polyhedron in two ways, according as one regards the point M as belonging to the one or to the other of the two lips of the cut. Let us superpose the two modes of subdivision, considering at once the edges arising from the one and from the other mode. We shall thus obtain a certain polyhedron, which I shall call P' ; one can arrange for it to remain homeomorphic to itself when the point M describes the whole cut OA_i (*vide infra*, § 5).

I shall call F'_k, B'_k, C'_k the faces, edges and vertices of P' . Each of the faces F of the first mode of subdivision will decompose into a certain number of faces F' , and the same will hold for each of the

faces F of the second mode; so that each face F' will belong to one of the faces F of the first mode, and to one only, and to one of the faces F of the second mode, and to one only.

Each of the edges B of each of the two modes of subdivision will decompose into a certain number of edges B' . Each edge B' will belong to at least one of the edges B of one of the two modes, and perhaps to one edge of each mode; but in no case will it belong to two different edges of the same mode.

Finally, the vertices C' will be the vertices C of the two modes, to which one must adjoin the points of intersection of the edges of the first mode with those of the second.

Likewise, we have seen that the surface S_0 which corresponds to the point O is decomposed into a polyhedron in q different ways. Let us superpose the q modes of subdivision; we shall obtain a polyhedron P'' whose faces, edges and vertices will be called F''_k, B''_k, C''_k . Each of the faces F corresponding to one of the q modes, as well as each of the faces F' corresponding to the polyhedron P'_i , which one obtains by regarding the point O as belonging to the cut OA_i , will be found decomposed into a certain number of faces F'' . Each face F'' will belong to one of the faces F' of the polyhedron P'_i , and to one only; to one of the faces F of each of the q modes of subdivision, and to one only.

Each of the edges B of the q modes, and each of the edges B' of the various polyhedra P'_i , will decompose into a certain number of edges B'' . Each edge B'' will belong to one of the edges B of one of the q modes, and to one of the edges B' of one of the polyhedra P'_i ; it may belong at once to two edges of two different modes, or to two edges B' of two different polyhedra P' , but not to two edges B of the same mode or to two edges B' of the same polyhedron.

The vertices C'' will consist of the vertices of the q modes, to which one must adjoin the points of intersection of the edges B belonging to different modes.

Nothing is to be changed as regards the manifolds of category 1; let us pass

to category $\alpha\beta$. Each of the cells $\alpha_i\beta_i F_k$ or $\alpha_i\beta_{i+1} F_k$ is going to be found decomposed into partial cells $\alpha_i\beta_i F'_k, \alpha_i\beta_{i+1} F'_k$; likewise, each of the faces $\alpha\beta B$ will be decomposed into faces $\alpha\beta B'$. To the edges $\alpha\beta C$

there will be adjoined other edges corresponding, as we have seen above, to the intersections of two edges B belonging to different modes. The set of these edges will constitute what I shall call the edges $\alpha\beta C'$.

One sees that the cells $\alpha_i\beta_i F'_k$ are identical to the cells $\alpha_i\beta_{i+1} F'_k$; likewise for the faces $\alpha_i\beta_i B'_k$ and $\alpha_i\beta_{i+1} B'_k$ and for the edges $\alpha_i\beta_i C'_k$ and $\alpha_i\beta_{i+1} C'_k$. But it is important to remark that the cell $\alpha_i\beta_i F_k$ decomposes into partial cells $\alpha_i\beta_i F'$ which are not, in general, the same as the partial cells into which the cell $\alpha_i\beta_{i+1} F_k$ decomposes. The same observation applies to the faces $\alpha_i\beta_i B_k$ and $\alpha_i\beta_{i+1} B_k$.

Let us pass to category α . When the point M comes to A_i , the two modes of decomposition of the surface S into a polyhedron P coincide; on the other hand, this polyhedron degenerates, as I have said, so that some of the manifolds $\alpha_i F'_k, \alpha_i B'_k, \alpha_i C'_k$ may disappear and some may coincide.

Let us pass to category β . We are going to have partial manifolds $\beta_i F''_k, \beta_i B''_k, \beta_i C''_k$.

The faces $\beta_1 F''_k, \beta_2 F''_k, \dots, \beta_q F''_k$ will be identical; but the face $\beta_1 F_k$ decomposes into partial faces $\beta F''$ which are not, in general, the same as those into which the face $\beta_2 F_k$, or the face $\beta_3 F_k$, etc., decompose. The same observation for the edges.

This being posed, I first make the following remark:

A manifold of category α will always be homologous to a sum of manifolds belonging to other categories.

Take, for example, the face $\alpha_i F'_k$; it belongs to the cell $\alpha_i\beta_i F'_k$, which admits in addition the face $\beta_i F'_k$ and those of the faces $\alpha_i\beta_i B'_h$ which correspond to edges B'_h belonging to the face F'_k of the polyhedron P' . If, then, one has the congruence (for this polyhedron P')

$$F'_k \equiv \Sigma \varepsilon_q B'_q,$$

the ε_q being numbers equal to $+1, -1$ or 0 , one will have the congruence

$$\alpha_i\beta_i F'_k \equiv \alpha_i F'_k - \beta_i F'_k + \Sigma \varepsilon_q \alpha_i\beta_i B'_q$$

and, consequently, the homology

$$\alpha_i F'_k \sim \beta_i F'_k - \Sigma \varepsilon_q \alpha_i\beta_i B'_q.$$

The manifolds which figure in the second member of this homology belonging to the categories β and $\alpha\beta$, the theorem is proved, and one would establish it in the same way for $\alpha_i B'_k$ and $\alpha_i C'_k$.

§ 2. — Cycles of three dimensions.

I pass to the search for the homologies and congruences between these manifolds. I begin with the following remark:

We can always suppose that our congruences contain no manifold of category α .

Let, indeed,

$$\Sigma \alpha_i A + H \equiv 0$$

be a congruence in which the $\alpha_i A$ are manifolds of p dimensions of category α (corresponding to a manifold A of the polyhedron P or P'), and H a combination of manifolds of p dimensions of the other categories.

We shall then have, on our polyhedron P or P' , the congruence

$$A \equiv \Sigma \varepsilon a,$$

where the ε are integers and the a manifolds having one dimension fewer than A . One will then have the congruence

$$\alpha_i \beta_i A \equiv \alpha_i A - \beta_i A + \Sigma \varepsilon \alpha_i \beta_i a,$$

whence the homology

$$\alpha_i A \smile \beta_i A - \Sigma \varepsilon \alpha_i \beta_i a.$$

Combining this homology with the congruence

$$\Sigma \alpha_i A + H \equiv 0,$$

we find the congruence

$$\Sigma \beta_i A - \Sigma \Sigma \varepsilon \alpha_i \beta_i a + H \equiv 0,$$

which no longer contains any manifold of category α . Q. E. D.

To obtain the homologies between the cells, it suffices to consider those which are deduced from the hypercells.

Let us suppose that on the polyhedron P one has the congruence

$$F_k \equiv \Sigma \varepsilon_q B_q,$$

the ε being equal to $+1$, -1 or 0 ; one will have, for the polyhedron of four dimensions, the congruence

$$F_k \equiv \Sigma \varepsilon_q B_q + \Sigma \alpha_i \beta_i F_k - \Sigma \alpha_i \beta_{i+1} F_k.$$

and, consequently, the homology

$$\Sigma \varepsilon_q B_q \smile \Sigma \alpha_i \beta_{i+1} F_k - \Sigma \alpha_i \beta_i F_k. \quad (1)$$

We shall recall, moreover, that $\alpha_i \beta_i F_k$, as well as $\alpha_i \beta_{i+1} F_k$, can be replaced by the sum of several partial cells $\alpha_i \beta_i F'$.

Whence a first consequence; let $\Sigma \zeta_q B_q$ be any combination of the cells B_q , the ζ being integers. I suppose that on the polyhedron P one has, between the corresponding edges B_q , the homology

$$\Sigma \zeta_q B_q \smile 0,$$

that is to say, that the set of these edges (each affected with the coefficient ζ_q) forms on the surface S a closed cycle capable of being reduced to a point by continuous deformation.

Then one will have on the polyhedron P the congruence

$$\Sigma \zeta_q B_q \equiv \Sigma \theta_k F_k,$$

the θ being integers; one will then have on the polyhedron of four dimensions the congruence

$$\Sigma \theta_k F_k \equiv \Sigma \zeta_q B_q + \Sigma \theta_k \alpha_i \beta_i F_k - \Sigma \theta_k \alpha_i \beta_{i+1} F_k,$$

whence the homology

$$\Sigma \zeta_q B_q \sim \Sigma \theta_k \alpha_i \beta_{i+1} F_k - \Sigma \theta_k \alpha_i \beta_i F_k.$$

If, then, a combination of edges B is homologous to zero on P , the corresponding combination of cells B will be homologous to a combination of cells of category $\alpha\beta$.

Let us now look for the congruences between the cells.

These congruences are of the form

$$\Sigma \zeta_q B_q + \Sigma \theta'_k \alpha_i \beta_i F'_k \equiv 0, \quad (2)$$

the ζ and the θ' being integers. *I say first that one will have on the polyhedron P :*

$$\Sigma \zeta_q B_q \equiv 0,$$

that is to say, that the set of the edges (affected with the coefficients ζ) must form one or several cycles on the surface S .

Let there be, indeed, on P the congruence

$$B_q \equiv \Sigma \varepsilon_h C_h.$$

We shall have on our polyhedron of four dimensions the congruence

$$B_q \equiv \Sigma \varepsilon_h C_h + H,$$

H denoting a combination of faces not belonging to category 1. One will have, on the other hand,

$$\alpha_i \beta_i F'_k \equiv H,$$

H still having the same signification; one deduces from this

$$\Sigma \zeta_q B_q + \Sigma \theta'_k \alpha_i \beta_i F'_k \equiv \Sigma \zeta_q \varepsilon_h C_h + H,$$

H still having the same signification.

As the second member must be identically null, one must have identically

$$\Sigma \zeta_q \varepsilon_h C_h = 0.$$

One will therefore have on the polyhedron P

$$\Sigma \zeta_q B_q \equiv \Sigma \zeta_q \varepsilon_h C_h = 0. \quad \text{Q. E. D.} \quad (3)$$

One must therefore have, by virtue of (2),

$$\Sigma \theta'_k \alpha_i \beta_i F'_k = \Sigma \zeta_q \alpha_i \beta_{i+1} B_q - \Sigma \zeta_q \alpha_i \beta_i B_q. \quad (4)$$

Let $S(M)$ be what the Riemann surface S becomes when the point y comes to M on the cut OA_i . Let MF'_k be the face F'_k of the polyhedron P' corresponding to this position of the point M ; let MP be the limit towards which the polyhedron P tends when the point y approaches M from the side $\alpha_i \beta_i$ of the cut. Let (MP) be the limit towards which this same polyhedron tends when the point y tends towards M from the side $\alpha_i \beta_{i+1}$ of the cut. Let MB_q and (MB_q) be the edges B_q of the two polyhedra MP and (MP) .

If we then take up again the congruence (4) and, in each of the manifolds which figure in it, keep only the points which belong at the same time to $S(M)$, we shall have a new congruence

$$\Sigma \theta' MF'_k \equiv \Sigma \zeta_q (MB_q) - \Sigma \zeta_q MB_q. \quad (5)$$

The expressions $\Sigma \zeta_q (MB_q)$ and $\Sigma \zeta_q MB_q$ represent two cycles drawn on the surface $S(M)$; let Ω'_i and Ω_i be these two cycles. From the congruence (5) one can derive the homology

$$\Omega'_i - \Omega_i \sim O, \quad (6)$$

⁴ which must hold on the surface $S(M)$.

Let us suppose that y , starting from the point M on the side $\alpha_i \beta_i$ of the cut, turns round the singular point A_i and returns to the point M on the other side of the cut. The cycle $\Sigma \zeta_q B_q$, being deformed in a continuous manner, will have as initial position $\Sigma \zeta_q MB_q = \Omega_i$ and as final position $\Sigma \zeta_q (MB_q) = \Omega'_i$.

The homology therefore expresses that Ω_i is homologous to its transform Ω'_i (by the transformation which the cycles of S undergo when y turns round the singular point A_i). The cycle $\Sigma \zeta_q B_q$ therefore remains homologous to itself as a result of this transformation, and also as a result of the analogous transformations corresponding to the other singular points. The cycles which enjoy this property may be called *invariant cycles*. We shall return to this notion further on. We see that to every congruence of the form (2) there corresponds an invariant cycle. *I say that, conversely, to every invariant cycle there corresponds a congruence of the form (2).*

Let, indeed, $\Sigma \zeta_q B_q$ be an invariant cycle. One will have on $S(M)$

$$\Sigma \zeta_q (MB_q) \sim \Sigma \zeta_q MB_q \quad (6)$$

and, consequently, one can find integers θ' such that

$$\Sigma \theta'_k MF'_k \equiv \Sigma \zeta_q (MB_q) - \Sigma \zeta_q MB_q. \quad (5)$$

⁴ *Editorial note:* The print sets a capital O here in place of the zero; 0 is meant.

One deduces from this the congruence

$$\Sigma\theta'_k\alpha_i\beta_iF'_k \equiv \Sigma\zeta_q\alpha_i\beta_{i+1}B_q - \Sigma\zeta_q\alpha_i\beta_iB_q + \Sigma\theta'_k\alpha_iF'_k - \Sigma\theta'_k\beta_iF'_k.$$

The sign Σ refers here to the index k , the index i being constant; but from this we can deduce

$$\Sigma\zeta_qB_q + \Sigma\Sigma\theta'_k\alpha_i\beta_iF'_k \equiv \Sigma\Sigma\theta'_k\alpha_iF'_k - \Sigma\Sigma\theta'_k\beta_iF'_k,$$

the double sign Σ then referring to the two indices i and k , for one evidently has

$$\Sigma\zeta_qB_q \equiv \Sigma\Sigma\zeta_q\alpha_i\beta_iB_q - \Sigma\Sigma\zeta_q\alpha\beta_{i+1}B_q.$$

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It remains for us to prove that one must have

$$\Sigma\theta'_k\alpha_iF'_k = 0, \quad \Sigma\Sigma\theta'_k\beta_iF'_k = 0.$$

Let us first see what concerns the first of these equalities.

Let us take up again the congruence (5) and make the point M tend towards A_i ; in the limit MF'_k will reduce to $\alpha_iF'_k$, and Ω_i will coincide with Ω'_i and with $\Sigma\zeta_q\alpha_iB_q$; the congruence (5) will therefore become

$$\Sigma\theta'_k\alpha_iF'_k \equiv 0. \tag{7}$$

Let us examine the signification of this congruence. Let us first suppose that for $y = A_i$ the surface S does not decompose, that is to say, that the curve

$$f(x, A_i, z) = 0$$

is indecomposable. Then the congruence can hold only under two hypotheses: either one has the identity

$$\Sigma\theta'_k\alpha_iF'_k = 0,$$

I mean that in the first member there can figure (with a coefficient θ' different from zero) only the faces $\alpha_iF'_k$ which disappear in consequence of the degeneration of the polyhedron P'_i , as I explained above.

Or else the combination $\Sigma\theta'_k\alpha_iF'_k$ will represent (once or several times) the entire Riemann surface; so that

$$\Sigma\theta'_k\alpha_iF'_k = nS.$$

But for the surface S corresponding to the point M we can write

$$S = \Sigma MF'_k,$$

since the set of the faces F'_k of the polyhedron P'_i must cover the entire surface S .

⁵*Editorial note:* The index of α in the last term did not print; α_i is expected, as in the preceding term.

One has, moreover,

$$\Sigma MP'_k \equiv 0.$$

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We shall therefore have the congruence

$$\Sigma(\theta'_k - n)MF'_k \equiv \Omega'_i - \Omega_i, \quad (5 \text{ bis})$$

and, on the other hand, when M comes to A_i ,

$$\Sigma\alpha_i F'_k = S,$$

and consequently

$$\Sigma(\theta'_k - n)\alpha_i F'_k = 0.$$

The second hypothesis is thus brought back to the first; for this it suffices to change θ'_k into $\theta'_k - n$, which is permitted, since the congruence (5) is thereby replaced by a congruence (5 *bis*) of the same form.

Let us now suppose that the curve $F(x, A_i, z)$ decomposes and, for example, that the corresponding Riemann surface decomposes into two partial surfaces S_1 and S_2 .

Then our congruence (7) could hold provided that one had

$$\Sigma\theta'_k\alpha_i F'_k = n_1 S_1 + n_2 S_2,$$

n_1 and n_2 being integers. That at least is what one might fear, but one can see in several ways that it will not be so.

The simplest is to reason as follows:

Let us start from the Riemann surface S_0 which corresponds to the point O ; let us make y vary in a continuous fashion from O to A_i following the cut OA_i ; the Riemann surface S will be deformed in a continuous manner while remaining homeomorphic to itself. Let $S(M)$ be the surface S corresponding to the point M . To each point of $S(M)$ one can make correspond a point of S_0 , since one can pass from $S(M)$ to S_0 by continuous deformation. Let us consider on $S(M)$ the two cycles Ω_i and Ω'_i ; to these two cycles there will correspond on S_0 two closed cycles, which I shall call U_i and U'_i .

When the point M traverses OA_i with a continuous motion, the two cycles U_i and U'_i will move with a continuous motion on the surface S_0 . When M is very near A_i , the two cycles Ω_i and Ω'_i and,

consequently, the two cycles U_i and U'_i are very near each other. When the point M comes to O , the two cycles Ω_i and U_i become identical and reduce to

$$\Sigma\zeta_q\beta_i B_q = \Omega_i^0;$$

the two cycles Ω'_i and U'_i become identical and reduce to

$$\Sigma\zeta_q\beta_{i+1} B_q = \Omega_{i+1}^0.$$

⁶ *Editorial note:* Printed MP'_k ; MF'_k is expected, as in the preceding display; an apparent misprint.

Let us suppose that the point M varies from A_i up to a certain point M_0 of the cut OA_i ; the two cycles U_i and U'_i , at first coincident, will separate and sweep out a certain region R of S_0 ; to this region there will correspond on $S(M_0)$ a certain region which will be formed of a certain number of faces of a corresponding polyhedron P'_i , since it is bounded by the two cycles Ω_i and Ω'_i , which are formed of edges of this polyhedron. We can write the congruence

$$\Sigma \theta'_k M F'_k \equiv \Omega'_i - \Omega_i, \quad (5)$$

where $\Sigma \theta'_k M_0 F'_k$ will represent precisely the region we have just defined.

This region reducing to zero when M comes to A_i , we shall have, in all cases,

$$\Sigma \theta'_k \alpha_i F'_k = 0. \quad (8)$$

We are now going to show that

$$\Sigma \Sigma \theta'_k \beta_i F'_k = 0,$$

and for this we are going to study the sum

$$\Sigma \theta'_k \beta_i F'_k.$$

From what precedes, it is nothing other than the region swept out on the surface S_0 by the two cycles U_i and U'_i when the point M varies from A_i to O ; it is bounded by the two cycles Ω_i^0 and Ω_{i+1}^0 .

One sees that we shall have

$$\begin{aligned} \Sigma \theta'_k \beta_1 F'_k &\equiv \Omega_2^0 - \Omega_1^0, \\ \Sigma \theta'_k \beta_2 F'_k &\equiv \Omega_3^0 - \Omega_2^0, \\ &\dots\dots\dots, \\ \Sigma \theta'_k \beta_q F'_k &\equiv \Omega_1^0 - \Omega_q^0, \end{aligned}$$

whence, by adding,

$$\Sigma \Sigma \theta'_k \beta_i F'_k \equiv 0.$$

This congruence shows us that the combination $\Sigma \Sigma \theta'_k \beta_i F'_k$ reduces to zero or to a certain number of times the surface S_0 .

Let us see whether this latter hypothesis can present itself. Let us suppose that y describes a closed contour while remaining infinitely near the cuts OA_i , describing first one of the lips of OA_1 , then the other lip, then the two lips of OA_2 , and so on, then finally the two lips of OA_q . Let us consider a cycle on the corresponding Riemann surface; this cycle will at first coincide with Ω_1^0 when y is at O on the first lip of OA_1 ; when y describes the first lip of OA_1 , this cycle will be deformed in a continuous manner, it will be our cycle Ω_1 , and the corresponding points on S_0 will form the cycle U_1 ; when the point y returns from A_1 to O by the second lip, this cycle will be nothing other than our cycle Ω'_1 , and the corresponding points on S_0 will form the cycle U_1^7 ; when y has

⁷Editorial note: Printed U_1 ; U'_1 is expected, the image of Ω'_1 ; an apparent misprint.

returned to O , this cycle will coincide with Ω_2^0 ; when y describes the two lips of OA_2 , the cycle will merge first with Ω_2 and then with Ω'_2 , and the corresponding points on S_0 will form the cycle U_2 , then the cycle U'_2 , and so on. The question is to know whether this moving cycle, which coincides successively with U_1 , with $U'_1, U_2, U'_2, \dots, U_q, U'_q$ and, in its initial as in its final position, coincides with Ω_1^0 , has described the entire surface S_0 .

To account for this, we must see why these cycles are deformed.

Let us consider the equation

$$f(x, y, z) = 0,$$

where y will be regarded as a constant; the singular points of z , regarded as a function of x , will be given by the equation

$$\frac{df}{dz} = 0.$$

Let us consider a cycle drawn on the Riemann surface. To this cycle there will correspond in the x -plane a certain contour which will enclose a certain number of these singular points. When y varies, these singular points will move, and if we do not want the cycle to pass through one of these singular points, we shall have to deform it so as to make it flee before these moving singular points. Whatever the displacements of these singular points, provided that two of them do not come to coincide, it will always be possible to deform the cycle continuously in such a way that it never passes through any of these points; one can even choose a certain number of fixed points and deform the cycle in such a way that it never passes through any of these singular points nor through any of the fixed points, provided that these singular points never coincide either with one another or with the fixed points.

When y varies, the singular points will move, and the corresponding points on S_0 will move likewise. They could coincide only if y came to one of the points A_i ; but we make y turn round these points, approaching them very closely but without reaching them. On the other hand, these points are going to describe lines, and we can find on S_0 a region ρ which will be crossed by none of these lines. It is the points of this region ρ that will play the role of the fixed points of which I was speaking just now. Then we can deform our cycle in such a way that, without ever passing through one of the singular points, it never enters the region ρ . It cannot, therefore, generate the entire surface S_0 . The hypothesis in question must be rejected, so that one will always have the identity

$$\Sigma \theta'_k \beta_i F'_k = 0. \quad (9)$$

To finish, we must remark that *there can be no congruences between cells of category $\alpha\beta$ alone*. Let, indeed,

$$\Sigma \theta'_k \alpha_i \beta_i F'_k \equiv 0$$

be such a congruence; we shall first find

$$\Sigma \theta'_k \alpha_i \beta_i F'_k \equiv \Sigma \theta'_k \varepsilon'_h \alpha_i \beta_i B'_h + \Sigma \theta'_k \alpha_i F'_k - \Sigma \theta'_k \beta_i F'_k;$$

so that one will have

$$\Sigma \theta'_k \varepsilon'_h \alpha_i \beta_i B'_h = 0, \quad \Sigma \theta'_k \alpha_i F'_k = 0, \quad \Sigma \theta'_k \beta_i F'_k = 0,$$

and, consequently, on the polyhedron P'_i (reasoning as we did to deduce 5 from 4),

$$\Sigma \theta'_k M F'_k \equiv 0.$$

The set of the faces $M F'_k$ (affected with the numerical coefficients θ'_k) of the polyhedron P'_i must therefore be congruent to zero, that is to say, form a closed surface, which can only be the entire surface of $S(M)$. The set $\Sigma \theta'_k \alpha_i F'_k$ will then represent the entire surface $S(A_i)$; one cannot, therefore, have

$$\Sigma \theta'_k \alpha_i F'_k = 0.$$

Therefore *our congruence is impossible*.

We know that, to obtain all the cycles of three dimensions, it suffices to look for the combinations of cells which are congruent to zero without being homologous to zero.

First, such a combination must contain cells of category 1, as we have just seen; let $\Sigma \zeta_q B_q$ be the set of these cells; the set of the corresponding edges must form a cycle on the surface S (it is the cycle Ω_i). This cycle must not be homologous to zero. If, indeed, this cycle were homologous to zero, the combination $\Sigma \zeta_q B_q$ would be homologous to a combination of cells of category $\alpha\beta$; one could therefore replace $\Sigma \zeta_q B_q$ by this combination in the first member of our congruence; this first member would then contain only cells of category $\alpha\beta$, which is impossible according to what we have just seen.

Finally, our cycle Ω_i must be *invariant*, that is to say, it must change into a homologous cycle Ω'_i when y turns round A_i . But when y turns round one of the singular points A_i , the cycles of the Riemann surface undergo one of the substitutions of the *Picard group*. The cycle Ω_i must therefore be invariant for the Picard group.

Thus, to every cycle of three dimensions of V there corresponds a cycle of the surface S invariant for the Picard group.

Conversely, let us consider a cycle invariant for the Picard group. If Ω_i is a position of this cycle on the polyhedron P'_i , and if Ω'_i is what Ω_i becomes when y has turned round A_i , one will have

$$\Omega_i \sim \Omega'_i.$$

One can find integers θ' so as to satisfy the congruence (5). The congruence (4) will likewise hold. But we have just seen that under these conditions the identities (8) and (9) hold, so that the congruence (4) can be written

$$\Sigma \Sigma \theta'_k \alpha_i \beta_i F'_k \equiv \Sigma \zeta_q \alpha_i \beta_{i+1} B_q - \Sigma \zeta_q \alpha_i \beta_i B_q,$$

whence

$$\Sigma \zeta_q B_q + \Sigma \Sigma \theta'_k \alpha_i \beta_i F'_k \equiv 0.$$

The first member of this congruence represents a cycle of three dimensions.

In summary, as many distinct invariant cycles as the Picard group admits, so many distinct cycles of three dimensions will the manifold V admit.

The best way to picture these cycles of three dimensions is to suppose that one has not merely

$$\Omega_i \sim \Omega'_i,$$

but identically

$$\Omega_i = \Omega'_i;$$

this is a supposition which we can always make, because of the

arbitrary fashion in which one can make our Riemann surfaces correspond point by point.

Under these conditions, one will give y all possible values.

To each value there will correspond a position of the cycle Ω_i , and, because of the invariance of this cycle, *to two infinitely near points situated on either side of one of the cuts there will correspond two infinitely⁸ little different positions of the cycle Ω_i .*

The various positions of this cycle will therefore generate a closed cycle of three dimensions.

§ 3. — Cycles of two dimensions.

To find all the cycles of two dimensions, it suffices to look for all the combinations of faces which are congruent to zero without being homologous to zero.

We can first suppose that this combination contains no face of category α ; for, according to what we established at the beginning of the preceding section, every face of category α is homologous to faces of the categories $\alpha\beta$ and β .

The question now is whether these combinations can contain faces of category 1. I first observe this: let C_1 and C_2 be any two vertices of the polyhedron P ; one can always pass from the one to the other by following certain edges of this polyhedron, so that we shall have on this polyhedron the congruence

$$C_1 - C_2 \equiv \Sigma \zeta_q B_q,$$

the second member representing the set of the edges by which one passes from C_1 to C_2 ; more generally, one can find integers ζ such that

$$\Sigma \varepsilon_k C_k \equiv \Sigma \zeta_q B_q, \tag{1}$$

provided that the ε are integers such that

$$\Sigma \varepsilon_k = 0;$$

for then $\Sigma \varepsilon_k C_k$ can be regarded as a sum of differences such as $C_1 - C_2$.

⁸*Editorial note:* Printed “infinies peu”; “infiniment peu” is expected; an apparent misprint.

Let us then consider on the manifold V the combination of faces $\Sigma\varepsilon_k C_k$, and let us suppose $\Sigma\varepsilon_k = 0$. We can then find integers ζ satisfying the congruence (1); there will then result on the manifold V

$$\Sigma\zeta_q B_q \equiv \Sigma\varepsilon_k C_k + \Sigma\zeta_q \alpha_i \beta_i B_q - \Sigma\zeta_q \alpha_i \beta_{i+1} B_q,$$

and, consequently,

$$\Sigma\varepsilon_k C_k \sim \Sigma\zeta_q \alpha_i \beta_{i+1} B_q - \Sigma\zeta_q \alpha_i \beta_i B_q,$$

which shows that the combination $\Sigma\varepsilon_k C_k$ is homologous to a combination of faces of category $\alpha\beta$.

Let us now suppose that one has a congruence of the form

$$\Sigma\varepsilon_k C_k + H = 0,$$

⁹ H representing a combination of faces of the categories $\alpha\beta$ and β . If one has $\Sigma\varepsilon_k = 0$, one can replace $\Sigma\varepsilon_k C_k$ in the first member by the combination of faces of the categories of β and $\alpha\beta$ which is homologous to it; and then this first member no longer contains faces of category 1.

If now we had two congruences of the same form

$$\Sigma\varepsilon_k C_k + H \equiv 0, \quad \Sigma\varepsilon'_k C_k + H' \equiv 0,$$

we could find two integers n and n' such that

$$n\Sigma\varepsilon_k + n'\Sigma\varepsilon'_k = 0,$$

and then the congruence

$$\Sigma(n\varepsilon_k + n'\varepsilon'_k)C_k + nH + n'H' \equiv 0,$$

which is a combination of the two preceding ones, could be brought back so as no longer to contain faces of category 1.

In summary, if there are congruences containing faces of category (1), there cannot be two distinct ones.

Let us therefore consider specially the congruences in which only faces of the categories β and $\alpha\beta$ figure. Let

$$\Sigma\theta'_k \alpha_i \beta_i B'_k + \Sigma\theta''_k \beta_i F''_k \equiv 0 \tag{2}$$

be one of these congruences.

Let us consider the points common to the manifolds which figure in this congruence and to the surface $S(M)$; let us reduce each of these manifolds to these common points; there will result

$$\Sigma\theta'_k M B'_k \equiv 0. \tag{3}$$

⁹ *Editorial note:* Printed = with two strokes; the text calls this relation a congruence and restates it below with $\equiv$.

Indeed, the point M being a point of the cut OA_i different from O , the surface $S(M)$ has no point in common with the faces of category β , for which y can take only the value O . Likewise, the surface $S(M)$ will have no point in common with the faces $\alpha_j\beta_jB'_k$ in which the index j is different from i , since for these faces y would have to be on the cut OA_j , whereas M is on the cut OA_i .

The congruence (3) signifies that, on the surface $S(M)$, the set of the edges B'_k of the polyhedron P'_i (affected with the coefficients θ') must form a closed cycle; let K_i be this cycle.

But let us observe that

$$\Sigma\theta'_k\alpha_i\beta_iB'_k \equiv \Sigma\theta'_k\alpha_iB'_k + H,$$

H being a set of edges of the categories β and $\alpha\beta$. Likewise

$$\Sigma\theta''_k\beta_iF''_k \equiv H',$$

H' being a combination of edges of category β .

One therefore has

$$\Sigma\theta'_k\alpha_i\beta_iB'_k + \Sigma\theta''_k\beta_iF''_k \equiv \Sigma\theta'_k\alpha_iB'_k + H + H',$$

so that the congruence (2) can hold only if one has identically

$$\Sigma\theta'_k\alpha_iB'_k = 0. \tag{4}$$

As there is no relation between the edges $\theta'_k\alpha_iB'_k$ belonging to different indices i , the identity (4) must hold when one gives the index i a determinate value and extends the summation only over the different values of the index k .

The identity (4) then signifies that, when the point M tends towards A_i , the cycle K_i tends to reduce to zero.

Indeed, when M reduces to A_i , the surface $S(M)$ degenerates and its genus diminishes; certain of its cycles therefore disappear. Let us see how this fact is connected with the study of the Picard group. Let S_i be the substitution of this group which corresponds to the singular point A_i ; it will change the cycle ω_h , for example, into

$$\omega'_h = m_1\omega_1 + m_2\omega_2 + \dots + m_{2p}\omega_{2p}.$$

Therefore, for $M = A_i$, one will have

$$\omega_h = \omega'_h;$$

which means that, for $M = A_i$, the cycle $\omega'_h - \omega_h$ disappears.

Shall we obtain in this way all the cycles which disappear for $M = A_i$, which I shall call *vanishing cycles*?

Mr. Picard has shown (vol. I, p. 82) that every algebraic surface can be brought by a birational transformation to have only ordinary singularities, that is to say, a double curve with triple points.

He then showed (p. 95) how one determines, for such a surface, the singular points A_i and the substitutions of the Picard group which correspond to each of them.

One thus sees that in this case there is only one vanishing cycle, which is generated precisely in the fashion I have just described.

If one wished to consider surfaces possessing more complicated singularities (for example, conical points), it might happen that there were others.

Let us suppose, for example, an ordinary surface admitting two singular points corresponding to the same substitution of the Picard group; if one makes this surface vary, and in the limit these two singular points coincide, the limiting surface will then admit a vanishing cycle not admitting this mode of generation.

If we then consider any vanishing cycle $\Sigma\theta'_k M B'_k$, we shall have the congruence

$$\Sigma\theta'_k \alpha_i \beta_i B'_k \equiv -\Sigma\theta'_k \beta_i B'_k. \quad (5)$$

To each singular point A_i let us make correspond in this way a vanishing cycle K_i , and consequently a congruence of the form (5). Let us add all these congruences; there will result

$$\Sigma\Sigma\theta'_k \alpha_i \beta_i B'_k \equiv -\Sigma\Sigma\theta'_k \beta_i B'_k.$$

¹⁰

Let us suppose that the set of the cycles K_i is homologous to zero, so that one has on the surface S

$$\Sigma K_i \sim 0,$$

one will have on S_0

$$\Sigma\Sigma\theta'_k \beta_i B'_k \sim 0,$$

that is to say, that one can find coefficients θ'' such that

$$\Sigma\Sigma\theta'_k \beta_i B'_k \equiv -\Sigma\theta''_k \beta_i F''_k :$$

¹¹ one will then have

$$\Sigma\Sigma\theta'_k \alpha_i \beta_i B'_k + \Sigma\theta''_k \beta_i F''_k \equiv 0. \quad (2)$$

Thus, to each combination of vanishing cycles K_i such that $\Sigma K_i \sim 0$ there will correspond a congruence of the form (2).

To the congruences thus obtained it is fitting to adjoin the following:

$$\Sigma\beta F''_k \equiv 0,$$

¹⁰ *Editorial note:* Strokes of the congruence sign did not print; $\equiv$ is expected, since this is the sum of the congruences (5).

¹¹ *Editorial note:* Strokes of the congruence sign did not print; $\equiv$ is expected, as in the congruences (5) and (2) of this page.

which represents a cycle of two dimensions formed of the entire surface S_0 . All the congruences of the form (2) will manifestly be combinations of those we have just obtained.

What, now, is the condition for two congruences of the form (2) to be distinct? or, in other words, which are the congruences of this form whose first member is homologous to zero?

To have all the homologies of the form

$$\Sigma \theta'_k \alpha_i \beta_i B'_k + \Sigma \theta''_k \beta_i F''_k \sim 0,$$

one must look for all the congruences between cells and faces of the form

$$\Sigma \theta'_k \alpha_i \beta_i B'_k + \Sigma \theta''_k \beta_i F''_k \equiv \Sigma \varepsilon_k B_k + \Sigma \zeta'_k \alpha_i \beta_i F'_k. \quad (6)$$

If we reduce all the manifolds which figure in the congruence (6) to their points in common with a surface S corresponding to a value of y not situated on one of the cuts, there results

$$\Sigma \varepsilon_k B_k \equiv 0,$$

which signifies that the set of edges $\Sigma \varepsilon_k B_k$ constitutes a closed cycle on the polyhedron P . Let $K(y)$ be this cycle.

Let $K(M_i)$ [or $K'(M_i)$] be the limit towards which this cycle tends when the point y approaches a point M_i belonging to the cut OA_i by the first lip of this cut (or by the second lip), so that $K(M_i)$ [or $K'(M_i)$] represents the set of the points common to the surface $S(M_i)$ ¹² and to $\Sigma \varepsilon_k \alpha_i \beta_i B_k$ (or to $\Sigma \varepsilon_k \alpha_i \beta_{i+1} B_k$).

One will then have (since $\Sigma \varepsilon_k B_k$ represents a closed cycle on the polyhedron P)

$$\Sigma \varepsilon_k B_k \equiv \Sigma \varepsilon_k \alpha_i \beta_i B_k - \Sigma \varepsilon_k \alpha_i \beta_{i+1} B_k; \quad (7)$$

¹³ whence

$$\Sigma \zeta'_k \alpha_i \beta_i F'_k \equiv \Sigma \theta'_k \alpha_i \beta_i B'_k + \Sigma \varepsilon_k \alpha_i \beta_{i+1} B_k - \Sigma \varepsilon_k \alpha_i \beta_i B_k + \Sigma \theta''_k \beta_i F''_k, \quad (8)$$

¹⁴

or, reducing all the manifolds to their points in common with the surface $S(M_i)$,

$$\Sigma \zeta'_k M_i F'_k \equiv \Sigma \theta'_k M_i B'_k + K'(M_i) - K(M_i); \quad (9)$$

that is to say, that one must have, on the polyhedron P'_i ,

$$K_i \sim K(M_i) - K'(M_i), \quad (10)$$

for $\Sigma \theta'_k M_i B'_k$ is nothing other than the cycle we called K_i above.

¹² *Editorial note:* The subscript i did not print; a gap stands after M where it belongs, as in $S(M_i)$ on p. 194.

¹³ *Editorial note:* Strokes of the congruence sign did not print fully; $\equiv$ is expected, as the text below calls (7) a congruence.

¹⁴ *Editorial note:* Strokes of the congruence sign did not print fully; $\equiv$ is expected, as p. 195 calls (8) a congruence.

If, moreover, the condition (10) is fulfilled, one can find numbers ζ' so as to satisfy the congruence (9); one will then have, making M_i tend towards A_i ,

$$\Sigma \zeta'_k \alpha_i F'_k \equiv \Sigma \theta'_k \alpha_i B'_k + K'(A_i) - K(A_i).$$

Now, by hypothesis, K_i is a vanishing cycle for the singular point A_i ; so that

$$\Sigma \theta'_k \alpha_i B'_k = 0.$$

Moreover,

$$K'(A_i) = K(A_i) = \Sigma \varepsilon_k \alpha_i B_k.$$

Therefore

$$\Sigma \zeta'_k \alpha_i F'_k \equiv 0;$$

which signifies that the set of the faces $\Sigma \zeta'_k \alpha_i P'_k$ ¹⁵ forms a closed surface; this can happen only if

$$\Sigma \zeta'_k \alpha_i F'_k = 0 \tag{11}$$

or if $\Sigma \zeta'_k \alpha_i F'_k$ represents the entire surface $S(K_i)$ ¹⁶ or one of the components of this surface, in the case where this surface is decomposable (see *supra*, p. 182).

If we leave aside this last case, which will not present itself with the surfaces admitting only ordinary singularities, to which Mr. Picard brings back all the others, and which could, moreover, be

treated as above (p. 182), one sees that one can always suppose

$$\Sigma \zeta'_k \alpha_i F'_k = nS(A_i),$$

n being an integer; or

$$\Sigma (\zeta'_k - n) \alpha_i F'_k = 0.$$

Now, in the congruence (9), one can replace ζ'_k by $\zeta'_k - n$ without the congruence ceasing to hold, for, the surface $S(M_i)$ being closed, one has

$$\Sigma M_i F'_k \equiv 0.$$

We can therefore always suppose that the identity (11) holds. There then results

$$\Sigma \zeta'_k \alpha_i \beta_i F'_k \equiv \Sigma \zeta'_k \alpha_i F'_k + \Sigma \theta'_k \alpha_i \beta_i B'_k + \Sigma \varepsilon_k \alpha_i \beta_{i+1} B_k - \Sigma \varepsilon_k \alpha_i \beta_i B_k - \Sigma \zeta'_k \beta_i F'_k.$$

Now, if we take account of the identity (11) and if we decompose the faces F' into faces F'' , we put

$$-\Sigma \zeta'_k \beta_i F'_k = \Sigma \theta''_k \beta_i F''_k;$$

we shall recover the congruence (8) and, by adding the congruence (7), we shall finally have the congruence (6).

¹⁵ *Editorial note:* Printed P'_k ; the faces F'_k are expected, as in the congruence just above; an apparent misprint.

¹⁶ *Editorial note:* Printed $S(K_i)$; the next page writes $nS(A_i)$ for this surface.

Thus, to each system of homologies such as (10) there corresponds a homology between the faces, and there are no others.

We verify likewise that the combination

$$\Sigma \beta_i F_k'',$$

which represents the entire Riemann surface S_0 and which, consequently, is congruent to zero, is not homologous to zero.

If, indeed, one had a congruence of the form (6), all the θ' being null, one would have to have a homology of the form (10), the cycles K_i being null; which means that the cycle $K(y)$ would be invariant for all the substitutions of the Picard group.

The cycles $K'(M_i)$ and $K(M_i)$ will then be nothing other than what

we called Ω'_i and Ω_i in the preceding section; we shall then recover the homology (5) of the preceding section, which will differ only in notation from the homology (9) of the present section; it suffices, indeed, in order to pass from the one to the other, to annul the θ' , to change $K'(M_i)$ into Ω'_i , $K(M_i)$ into Ω_i , and to write θ'_k instead of ζ'_k . To adopt the notations of the preceding section, one must finally write $\zeta_q B_q$ instead of $\varepsilon_k B_k$.

In this case we shall have between our cells the congruence (2) of the preceding section, which is written

$$\Sigma \zeta_q B_q + \Sigma \theta'_k \alpha_i \beta_i F'_k \equiv 0,$$

or, returning to the notations of the present section,

$$\Sigma \varepsilon_k B_k + \Sigma \zeta'_k \alpha_i \beta_i F'_k \equiv 0.$$

This shows that the first member of (6) must be identically null, that is to say, that not only the θ' but also the θ'' must be null.

Thus every homology between the faces $\beta F''$ reduces to an identity, and, in particular, one does not have

$$\Sigma \beta_i F_k'' \sim 0. \quad \text{Q. E. D.}$$

Before concluding, I must still examine the congruences into which cells of category 1 enter. We have just seen that there cannot be more than one such congruence, or rather that, if there were two such congruences, they would not be distinct, and one could pass from the one to the other by adding a homology.

Let us see whether such a congruence exists,

$$\Sigma \varepsilon_k C_k + H \equiv 0, \tag{12}$$

¹⁷ where H is a combination of faces of the categories $\alpha\beta$ and β . First, we must suppose that $\Sigma \varepsilon_k$ is not null, otherwise one could bring it back to one of the congruences studied above.

¹⁷ *Editorial note:* Strokes of the congruence sign did not print; $\equiv$ is expected, as the text calls (12) a congruence.

I add that, if there exists a congruence of this form (12) in which $\Sigma\varepsilon_k$ is not null, this congruence is certainly distinct from the preceding ones, for there cannot be a homology of the form

$$\Sigma\varepsilon_k C_k + H \sim 0 \quad (13)$$

without $\Sigma\varepsilon_k$ being null.

Does there, then, exist a congruence of the form (12)? To prove it without exposing myself to a discussion which would be rather long without being difficult, I shall suppose that among the vertices of the polyhedron P there figure m (which I shall call $C_1, C_2, \dots, C_m$), if m is the degree in z of the equation $f(x, y, z) = 0$, and which correspond to a given value of x , for example $x = x_0$ (see *infra*, § 5).

Then the combination $C_1 + C_2 + \dots + C_m$, which I shall write for short as ΣC_k , is nothing other than the Riemann surface represented by the equation between y and z

$$f(x_0, y, z) = 0.$$

One then has

$$\Sigma C_k \equiv \Sigma C_k \alpha_i \beta_i - \Sigma C_k \alpha_i \beta_{i+1} :$$

the m vertices $C_1, C_2, \dots, C_m$ merely being interchanged when one passes from one of the lips of OA_i to the other by turning round A_i , we shall have

$$\Sigma C_k \alpha_i \beta_i = \Sigma C_k \alpha_i \beta_{i+1}$$

and, consequently,

$$\Sigma C_k \equiv 0.$$

This congruence is indeed of the form (12), and

$$\Sigma\varepsilon_k = m \geq 0.$$

We therefore have, first, two singular cycles of two dimensions, which are the two Riemann surfaces corresponding, the one to $y = 0$ and the other to $x = x_0$.

To form the other cycles of two dimensions, it suffices to consider

q cycles

$$K_1, \quad K_2, \quad \dots, \quad K_q,$$

corresponding to the q singular points

$$A_1, \quad A_2, \quad \dots, \quad A_q,$$

each of them being vanishing with respect to the singular point which corresponds to it; these cycles must, moreover, satisfy, on the polyhedron P , the condition

$$\Sigma K_i \sim 0.$$

Two systems of cycles

$$\begin{array}{cccc} K_1, & K_2, & \dots, & K_q, \\ K'_1, & K'_2, & \dots, & K' \end{array}$$

¹⁸ will give us two distinct cycles of two dimensions, unless one has (on P)

$$K'_i - K_i \smile K'(M_i) - K(M_i),$$

$K(y)$ being any cycle of the polyhedron P .

If we denote by U_i and U'_i what the cycles K_i and K'_i become when the point M_i comes to O , and by Ω_i what the cycle $K(y)$ becomes when the point y tends towards O in the angle $A_{i-1}OA_i$, then this signifies that we must have on S_0

$$\Sigma U_i \smile 0, \quad \Sigma U'_i \smile 0,$$

and that we must not have, if we want two distinct cycles,

$$U'_i - U_i \smile \Omega_{i+1} - \Omega_i.$$

Let us see how many distinct cycles we shall obtain in this way, and for that how many distinct congruences of the indicated form we shall obtain; and from this we shall subtract the number of distinct homologies.

Let us form the Table of the vanishing cycles relative to the different singular points A_i . We shall distinguish two sorts of them:

1 $\bar{x}$ Those which are of the form $\Omega' - \Omega$, Ω' being the transform of the cycle Ω by the substitution of the Picard group which corresponds to A_i . These are the only ones which exist in general.

2 $\bar{x}$ The vanishing cycles of the second sort will be those which are neither of this form nor a combination of cycles of this form.

Evidently, for each of the singular points, we must reduce the number of these vanishing cycles as far as possible; we must therefore enter in our Table only linearly independent cycles. If, then, these cycles are

$$\begin{aligned} m_{1,1}\omega_1 + m_{1,2} + \dots + m_{1,2p}\omega_{2p}, \\ \dots\dots\dots, \\ m_{k,1}\omega_1 + m_{k,2} + \dots + m_{k,2p}\omega_{2p}, \end{aligned}$$

¹⁹ the Table of the integer coefficients m will have $2p$ columns and k rows ($k < 2p$), and the determinants formed by suppressing in our Table any $2p - k$ columns must not all be null at once.

Let us now unite the Tables relative to the q singular points; the total Table will have $2p$ columns and $\Sigma k = \mu$ rows. If the determinants formed by suppressing in this Table $\mu - 2p + r$ rows and r columns are all null, but the determinants formed by suppressing $p - 2p + r + 1$ ²⁰ rows and $r + 1$ columns are not all null, the required number of distinct congruences will be $\mu - 2p + r + 1$ (or $\mu - 2p$ if the determinants formed by suppressing $\mu - 2p$ rows are not all null).

¹⁸ Editorial note: The subscript of the last entry did not print; K'_q is expected.

¹⁹ Editorial note: The factor ω_2 after $m_{1,2}$ and after $m_{k,2}$ is missing in the print; $m_{1,2}\omega_2$ and $m_{k,2}\omega_2$ are expected.

²⁰ Editorial note: Printed $p - 2p + r + 1$; $\mu - 2p + r + 1$ is expected, matching the $\mu - 2p + r$ lines of the preceding clause; an apparent misprint.

To have the number of distinct homologies, one must look for the number of combinations of cycles $U_1, U_2, \dots, U_q$ such that one has

$$U_i \sim \Omega_{i+1} - \Omega_i.$$

Each cycle Ω will evidently give rise to such a combination; but if two cycles Ω and Ω' differ only by a cycle

invariant with respect to the Picard group, they will give rise to the same combination. The number of homologies is therefore equal to the total number of cycles, namely $2p$, less the number of distinct invariant cycles, which I call n .

The number of distinct cycles of two dimensions (*including the two singular cycles*) will therefore be

$$2 + (\mu + 2p + 1) - (n - 2p).$$

21

Let us see how these non-singular cycles of two dimensions can be generated, restricting ourselves to the most general case, that is to say, to the one in which there is no vanishing cycle of the second sort. In this case one has

$$K_i = \Gamma'_i - \Gamma_i,$$

Γ_i being a cycle and Γ'_i its transform by the substitution which corresponds to A_i .

Let us therefore describe in the y -plane loops $L_1, L_2, \dots, L_q$, starting from the point O and ending there, and enclosing respectively the singular points $A_1, A_2, \dots, A_q$.

Let Γ_i be a cycle of the surface S_0 ; when y , starting from the point O , describes the loop L_i , the surface S and the cycle will be deformed; when y returns to the point O , this cycle will have become the cycle Γ'_i of S_0 ; in its motion it will have generated a surface σ_i bounded by the two closed curves Γ_i and Γ'_i .

If, then, one has

$$\Gamma_1 + \Gamma_2 + \dots + \Gamma_q = \Gamma'_1 + \Gamma'_2 + \dots + \Gamma'_q,$$

the set of the surfaces $\sigma_1 + \sigma_2 + \dots + \sigma_q$ will form a closed surface. This will be our cycle of two dimensions.

§ 4. — Cycles of one dimension.

The problem of the cycles of one dimension has been entirely solved by Mr. Picard; I shall therefore only have to translate Mr. Picard's reasoning into our notations.

²¹*Editorial note:* As printed. The text gives $\mu - 2p + r + 1$ distinct congruences on p. 199 and $2p$ minus n homologies above; the signs in this count do not match those expressions.

Let us look for the congruences between the edges. As we saw at the beginning of section 2, we can always suppose that such a congruence contains no edges of category α . Our congruence must therefore be of the form

$$\Sigma \theta'_k \alpha_i \beta_i C'_k + \Sigma \theta''_k \beta_i B''_k \equiv 0. \quad (1)$$

22

Let us observe that

$$\begin{aligned} \Sigma \theta'_k \alpha_i \beta_i C'_k &\equiv \Sigma \theta'_k \alpha_i C'_k + H, \\ \Sigma \theta''_k \beta_i B''_k &\equiv H' \end{aligned}$$

²³ H and H' being a set of vertices of category β . There therefore results

$$\Sigma \theta'_k \alpha_i C'_k + H + H' = 0,$$

and, as the vertices $\alpha' C'_k$ ²⁴ can cancel neither with the vertices of category β nor with those of category α in which the index of α is different from i , one must have

$$\Sigma \theta'_k \alpha_i C'_k = 0, \quad (2)$$

the summation being extended to all the vertices $\alpha_i C'_k$ belonging to one and the same index i , but to the various indices k .

What does this identity (2) signify? When, y coming to A_i , the polyhedron P'_i degenerates, several vertices may coincide, but none can disappear (whereas there may be disappearance of edges or of faces). The algebraic sum of all the coefficients θ'_k relative to the various vertices $\alpha_i C'_k$ which coincide in a single one must therefore be null; therefore the sum of all the θ'_k is null:

$$\Sigma \theta'_k = 0. \quad (3)$$

Because of the relation (3), we can find on the polyhedron P'_i a combination of edges $\Sigma \zeta'_k B'_k$ such that

$$\Sigma \zeta'_k B'_k \equiv \Sigma \theta'_k C'_k. \quad (4)$$

One will then have

$$\Sigma \zeta'_k \alpha_i \beta_i B'_k \equiv \Sigma \zeta'_k \alpha_i B'_k - \Sigma \zeta'_k \beta_i B'_k + \Sigma \theta'_k \alpha_i \beta_i C'_k. \quad (5)$$

Moreover, when y comes to A_i , the edges and the vertices B'_k and C'_k of the polyhedron P'_i become the edges and the vertices $\alpha_i B'_k$ and $\alpha_i C'_k$; we therefore have the congruence

$$\Sigma \zeta'_k \alpha_i B'_k \equiv \Sigma \theta'_k \alpha_i C'_k$$

²² *Editorial note:* Strokes of the congruence sign did not print fully; $\equiv$ is expected, as the text calls (1) a congruence.

²³ *Editorial note:* In the second line the strokes of the congruence sign did not print; $\equiv$ is expected, as in the first line.

²⁴ *Editorial note:* Printed $\alpha' C'_k$; $\alpha_i C'_k$ is expected, as in the relations around it.

or, because of (2),

$$\Sigma \zeta'_k \alpha_i B'_k \equiv 0.$$

This congruence signifies that the combination $\Sigma \zeta'_k \alpha_i B'_k$ forms a closed cycle on the Riemann surface $S(A_i)$. But when, y coming to A_i , the Riemann surface degenerates, cycles may disappear, but it never happens that new cycles appear. Therefore, to the cycle $\Sigma \zeta'_k \alpha_i B'_k$ there will correspond on the surface $S(M_i)$ at least one closed cycle $\Sigma \varepsilon'_k M_i B'_k$, reducing to $\Sigma \zeta'_k \alpha_i B'_k$ for $M_i = \alpha_i$, so that one has

$$\begin{aligned} \Sigma \varepsilon'_k M_i B'_k &\equiv 0, \\ \Sigma \varepsilon'_k \alpha_i B'_k &= \Sigma \zeta'_k \alpha_i B'_k. \end{aligned} \tag{6}$$

If we then replace ζ'_k by $\zeta'_k - \varepsilon'_k$, the congruence (4) will still subsist because of (6); the congruence (5) will likewise still be true, and one will have

$$\Sigma (\zeta'_k - \varepsilon'_k) \alpha_i B'_k = 0.$$

We can therefore always suppose that the ζ'_k have been chosen in such a way that

$$\Sigma \zeta'_k \alpha_i B'_k = 0. \tag{7}$$

If we bring the congruence (5) together with the identity (7), we shall have the homology

$$\Sigma \theta'_k \alpha_i \beta_i C'_k \sim \Sigma \zeta'_k \beta_i B'_k \tag{8}$$

and, adding this homology to (1), there results

$$\Sigma \zeta'_k \beta_i B'_k + \Sigma \theta''_k \beta_i B''_k \equiv 0,$$

in which only edges of category β now figure.

We can therefore always suppose that our congruence (1) contains only edges of category β . This is Picard's theorem, according to which a cycle of one dimension can always be brought into a position such that y is constant all along this cycle.

To each of the closed cycles of the surface S_0 there therefore corresponds a congruence of the form (1), but all these congruences are not distinct. This is what Mr. Picard has shown. Let $\omega_1, \omega_2, \dots, \omega_{2p}$ be the $2p$ cycles of S_0 , and let us suppose that a substitution S_h of the Picard group changes ω_i into

$$m_1 \omega_1 + m_2 \omega_2 + \dots + m_{2p} \omega_{2p},$$

one will have the homology

$$\omega_i \sim m_1 \omega_1 + m_2 \omega_2 + \dots + m_{2p} \omega_{2p}. \tag{9}$$

These homologies reduce the number of the cycles of one dimension, and Mr. Picard has even shown that, for the most general algebraic surface, this number reduces to zero.

Let us call *subsisting cycles* those which are not a linear combination of various cycles vanishing with respect to various singular points A_i . They are

those which remain distinct when one takes account of the homologies (9). And, indeed, the homology (9) expresses that the cycle

$$\omega_i - \Sigma m_k \omega_k$$

is vanishing with respect to the singular point which corresponds to the substitution S_h of the Picard group.

There are therefore as many cycles of one dimension as subsisting cycles. We have seen, on the other hand, that there are as many cycles of three dimensions as invariant cycles.

Now, according to the fundamental theorem on the Betti numbers, there must be as many cycles of one dimension as cycles of three dimensions.

There must therefore be as many subsisting cycles as invariant cycles.

This is what we are going to verify.

This verification is easy if there are no vanishing cycles of the second sort.

Let us recall, indeed, that the Picard group is of a particular form.

Let $\omega_1, \omega_2, \dots, \omega_{2p}$ be the fundamental cycles, and let us consider the bilinear form

$$\Phi = \omega'_2 \omega_1 - \omega'_1 \omega_2 + \omega'_4 \omega_3 - \omega_3 \omega_4 + \dots + \omega'_{2p} \omega_{2p-1} - \omega'_{2p-1} \omega_{2p}.$$

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Provided that the fundamental cycles have been suitably chosen, if one subjects the ω to one of the linear substitutions of the Picard group, and at the same time subjects the ω' to this same linear substitution, the form Φ remains unaltered. On the other hand, the number of the subsisting cycles will be the same as that of the distinct solutions of the system (A) of linear equations obtained by equating each fundamental cycle to its transform by each of the Picard substitutions.

Let, then, $\Sigma m_i \omega_i$ be an invariant cycle; as the expression $\Sigma m_i \omega_i$ can be assimilated to the linear form Φ by putting

$$\omega'_2 = m_1, \quad \omega'_1 = -m_2, \quad \omega'_3 = m_4, \quad \omega'_4 = -m_3, \quad \dots;$$

²⁶ as, on the other hand, $\Sigma m_i \omega_i$ changes into $\Sigma m_i \omega_i$ when the ω undergo one of the linear substitutions of the Picard group, we must conclude that the system of values

$$-m_2, \quad m_1, \quad -m_3, \quad m_4, \quad \dots$$

is its own transform by this linear substitution. It is therefore a solution of the system (A) of which we have just spoken.

One sees, moreover, that to two or several linearly

independent invariant cycles there will thus correspond two or several linearly independent solutions of the system (A), and conversely.

²⁵ *Editorial note:* The prime on the fourth term did not print; $\omega'_3 \omega_4$ is expected from the pattern of the form.

²⁶ *Editorial note:* As printed, these values for ω'_3 and ω'_4 , and the list of values below, do not agree with the form Φ above, whose terms $\omega'_4 \omega_3 - \omega'_3 \omega_4$ would give $\omega'_4 = m_3, \omega'_3 = -m_4$.

There will therefore be as many subsisting cycles as invariant cycles. Q. E. D.
What will happen, now, if there are vanishing cycles of the second sort?

I have already said that this case cannot present itself for those surfaces with ordinary singularities to which Mr. Picard has brought back all the others (vol. I, p. 85). It is true that it could occur for other surfaces if one wished to study them without first subjecting them to Mr. Picard's transformation; but these surfaces would present a singular point of a special nature, and the manifold V of four dimensions generated by this surface would itself present a singular point.

Now, the general theorems which occupy us, and which we proved in the *Analysis Situs* and its supplements, are not applicable to manifolds V presenting singular points; they would cease, in general, to be true for these manifolds, unless one made special conventions.

The very question whether a vanishing cycle of the second sort is to be regarded as homologous to zero depends, moreover, on the equally legitimate conventions that one can make. It is true that such a cycle serves as the complete boundary of a manifold of two dimensions forming part of V ; but on this manifold there lies a singular point of V .

To make the resulting difficulty understood, let us take a much simpler example; let us imagine in ordinary space a surface presenting a conical point, or, more simply still, a cone of revolution with its prolongation. Let S be the vertex of the cone, C one of the circles of this cone. In one sense, the circle C is the complete boundary of a region of this cone, the one comprised between the circle C and the vertex S . But, on the other hand, a line drawn on the cone can leave this region without crossing C if, in passing through the vertex S , it passes from one nappe of the cone to the other.

Under these conditions, it will seem preferable to leave aside these singular cases and to confine ourselves to those surfaces with ordinary singularities to which all the others can be brought back.

§ 5. — Various remarks.

In the course of the proofs, we have been led to make various hypotheses on the subject of our polyhedra. Let us recall them summarily:

1^{re} We supposed (*see* p. 174) that the polyhedron which we called P' remained homeomorphic to itself when y followed the cut OA_i from A_i to O .

2^{re} We compared (*see* p. 182) the Riemann surface S_0 , which corresponds to the point O , with the Riemann surface $S(M)$, which corresponds to the point M , and we said that one could make these two surfaces correspond point by point. We made use of this correspondence to define the cycles U_i and U'_i which correspond on S_0 to Ω_i and Ω'_i .

3^{re} We then admitted (p. 183) that, when the point M varies from A_i to M_0 , the two cycles U_i and U'_i , at first coincident, sweep out a certain region R of S_0 , to which there corresponds on $S(M_0)$ the region $\Sigma\theta'_k M_0 F'_k$.

4^{re} We admitted that, when the point M describes successively the two lips of the cut OA_1 , then the two lips of $OA_2, \dots$, and finally the two lips of the cut

OA_q , the moving cycle U_i (which returns to its initial situation Ω_1^0 after having occupied a continuous series of successive situations) does not sweep out the entire surface S_0 .

5^{re} We said (p. 191) that, if the surface $f(x, y, z) = 0$ has been brought, by Mr. Picard's procedures, to possess only ordinary singularities, to each singular point A_i there corresponds a single vanishing cycle.

6^{re} We supposed (p. 197) that among the vertices of P there figure m which correspond to a constant value of x , for example $x = 0$.

The legitimacy of these hypotheses being almost evident, I did not wish to interrupt the reasoning of the preceding sections

in order to give an explicit proof of them. I could, moreover, have done so only by particularizing the polyhedron P , that is to say, by making particular hypotheses about the manner in which the Riemann surface S is subdivided into a polyhedron.

I now think it useful to return to these different points and to give the proof, adopting any one of these particular hypotheses on P .

One could, for example, construct the polyhedron P in the following fashion:

Let us begin by reducing the surface $f = 0$ to have only ordinary singularities.

Let us give y any value, and consider the corresponding Riemann surface S . This surface will be composed of m sheets laid on the x -plane (if the equation $f = 0$ is of degree m in z).

Let us mark on the x -plane the origin O and the singular points corresponding to the equations

$$f = 0, \quad \frac{df}{dz} = 0.$$

Let n be the number of these singular points; let $B_1, B_2, \dots, B_n$ be these singular points. Let us join the point O to the n singular points $B_1, B_2, \dots, B_n$ by n cuts $OB_1, OB_2, \dots, OB_n$ which do not cut one another and which follow one another round the point O in the order indicated above.

One will obtain the Riemann surface in the ordinary fashion by joining the first lip of one of the cuts on one of the sheets to the second lip of this same cut on another sheet. By the drawing of these cuts, the Riemann surface will thus be subdivided into a polyhedron, which will be our polyhedron P .

One sees that this polyhedron has m faces (which are the F_k) and that each of these faces is a polygon of $2n$ sides.

What happens now when one makes y vary? The singular points B are going to move; the cuts OB will be deformed, and we shall suppose that they are deformed in such a way as never to cut one another and always to follow one another in the same order round O . When y describes a small closed contour, this deformation can be made in such a way

that the cuts return to their original positions, unless there is a singular point inside the contour.

The possible singular points are of two sorts:

1st First, those which correspond to the cases in which two of the singular points B are interchanged (a reasoning of Mr. Picard shows that, if the surface

$f = 0$ has only ordinary singularities, this can happen only if the plane $y = \text{const.}$ is tangent to the surface $f = 0$);

2 $\tilde{x}$ Then those which correspond to the cases in which one of the singular points B comes to O .

The singular points of the second sort are not essential, and I could have made them disappear had I not thought it more advantageous to keep them.

I denote all these singular points by $A_1, A_2, \dots, A_q$, and I draw in the y -plane the cuts $OA_1, OA_2, \dots, OA_q$.

As long as y does not cross these cuts OA , the cuts OB can be deformed in such a way as never to cut one another, and consequently in such a way that P remains homeomorphic to itself, and at the same time in such a way that, if y describes a closed contour, these cuts OB return to their initial positions.

Let us now compare the configurations of the cuts OB when y is at two infinitely near points on the two lips of a cut OA_i .

Let us first suppose that A_i is a singular point of the first sort. If one supposes that the surface $f = 0$ has only ordinary singularities and that, consequently, the plane $y = A_i$ is tangent to $f = 0$ at an ordinary point, one sees that, when y turns round A_i , two of the singular points, for example B_1 and B_3 , are interchanged. Moreover, if the point B_1 permutes two of the sheets of the Riemann surface, the point B_3 which is interchanged with it will permute the same two sheets of this surface. I shall call these two sheets *the first and the second sheets of the surface*.

When y , having turned round A_i , returns infinitely near its original position, but on the other lip of the cut, one can suppose that the cuts OB have returned to their original situation, with the exception of the cuts OB_1 and OB_3 . These last two cuts,

which originally occupied the lines OB_1 and OB_3 drawn solid in the figure, will finally occupy the lines OB_3 and OB_1 drawn dotted in the figure.

One sees that the Riemann surface can be subdivided in two ways into a polyhedron P ; the two modes of subdivision differ from each other in that the solid lines OB_1 and OB_3 are replaced by the dotted lines OB_3 and OB_1 .

If one superposes the two modes of subdivision, one will have the polyhedron which I have called P' . One sees that two of the m faces of P , those which correspond to the first two sheets, have been subdivided into three partial faces, designated in the figure by the letters (α) , (β) , (γ) .

When y describes the cut OA_i , the points B will move in a continuous manner, without ever coinciding either with one another or with the point O (except, of course, when y comes to A_i). It follows that one can deform all our cuts OB (those drawn solid as well as those drawn dotted) while preventing them from ever cutting one another. This means that the polyhedron P' will remain constantly homeomorphic to itself. To say that P' remains homeomorphic to itself is to say that one can make the surface $S(M)$ correspond point by point to another surface $S(M')$ corresponding to another position M' of y on OA_i , and in particular to S_0 . It will be remarked that the correspondence can be made in such a way that to a point at infinity on $S(M)$ there corresponds a point at infinity on $S(M')$.

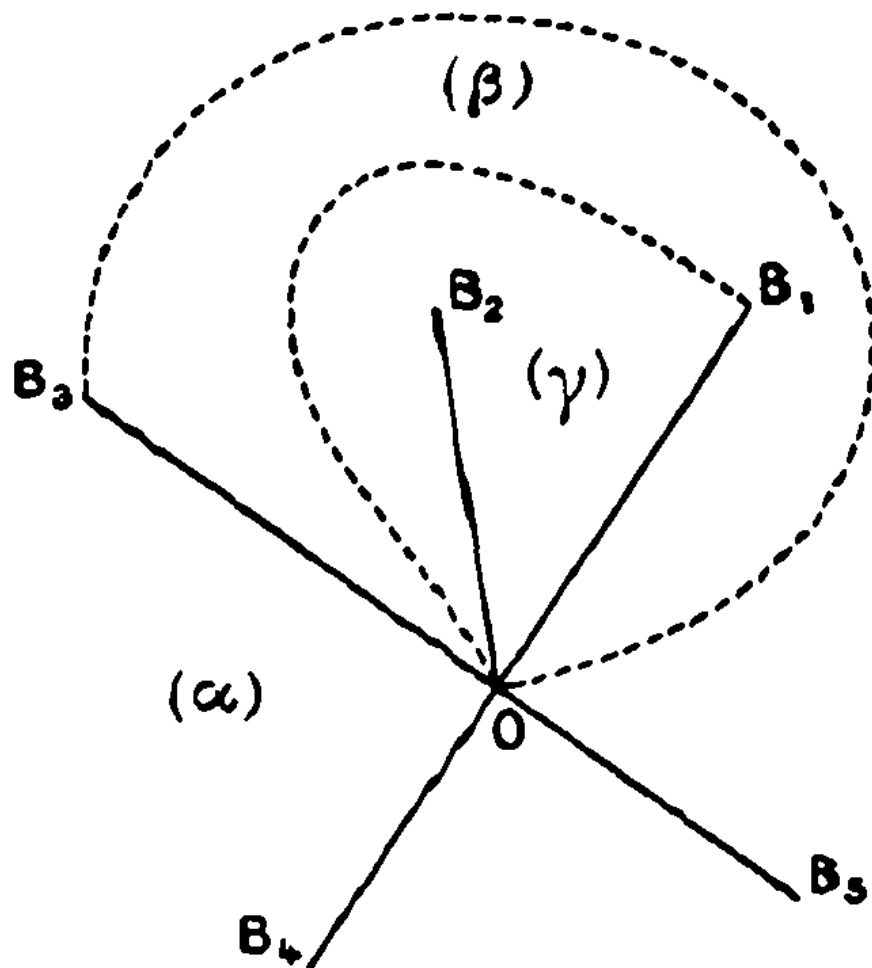


Figure 1: Fig. 1.

When y comes to A_i , the polyhedron P degenerates: the two points B_1 and B_3 coincide. The same holds for the solid cut OB_1 with the dotted OB_3 , as well as for the dotted OB_1 with the solid OB_3 . The partial polygon marked (β) in the figure then disappears.

To go further, let us remark that there are two cuts which project along the solid line OB_1 ; when one follows the first (which we shall call B_1G) from O to B_1 , one has the first sheet on the left and the second on the right; when one follows the second (which we shall call B_1D), one has the first sheet on the right and the second on the left. We shall define B_3G and B_3D in the same way. If, instead of the solid line OB_1 , we consider the dotted line OB_1 , we shall likewise obtain two cuts, which I shall call B'_1G and B'_1D ; and I shall again define in the same way B'_3G and B'_3D .

We see at once that, when y turns round A_i , the cuts B_1G and B'_3G , B_1D and B'_3D , B_3G and B'_1G , B_3D and B'_1D are permuted; and that, for $y = A_i$, B_1G and B'_3G , ... coincide.

I shall denote by β_1 that one of the partial polygons β which belongs to the first sheet if one adopts the first subdivision, the one which corresponds to the solid lines; the other polygon β will be called β_2 .

One then has the congruences

$$\begin{aligned}\beta_1 &\equiv B'_1D - B_1D + B'_3D - B_3D, \\ \beta_2 &\equiv B'_1G - B_1G + B'_3G - B_3G,\end{aligned}$$

which tell us which of our cuts serve as boundaries to β_1 and to β_2 .

Let us consider the combination

$$\omega = B_3D - B'_1D - B_3G + B'_1G.$$

It is a cycle of our Riemann surface; when y turns round A_i , it changes into

$$B'_1D - B_3D - B'_1G + B_3G,$$

that is to say, into $-\omega$. It is therefore a vanishing cycle. It is easy to see that there is no other.

Let us now take up again the cycles which we called Ω_i and Ω'_i in section 2. Let

$$\Omega_i = \zeta_1 B_1D + \zeta_1 B_1G + \zeta_3 B_3D + \zeta_1 B_3G + H,$$

²⁷ the ζ being integer coefficients and H a combination of other edges of P_1 ²⁸. As the cycle Ω_i must be closed, one must have

$$\zeta_1 + \zeta_2 = \zeta_3 + \zeta_4 = 0,$$

²⁷ *Editorial note:* The second and fourth coefficients are printed ζ_1 ; the condition $\zeta_1 + \zeta_2 = \zeta_3 + \zeta_4 = 0$ below and the display for Ω'_i require ζ_2 and ζ_4 ; apparent misprints, the fourth possibly a worn 4.

²⁸ *Editorial note:* Printed P_1 , a symbol defined nowhere in the paper; the polyhedron is P , as in "les arêtes de P " a few lines below; an apparent misprint.

since the vertex B_1 , for example, belongs only to the two edges B_1D and B_1G . We shall then have

$$\Omega'_i = \zeta_1 B'_3 D + \zeta_2 B'_3 G + \zeta_3 B'_1 D + \zeta_4 B'_3 G + H.$$

²⁹ For the edges of P other than B_1D, B_1G, B_3D, B_3G are not altered when y turns round A_i , that is to say, H is not altered. One will therefore have

$$\begin{aligned} \Omega_i - \Omega'_i &= \zeta_1(B_1D - B_1G - B'_3D + B'_3G) \\ &\quad + \zeta_3(B_3D - B_3G - B'_1D + B'_1G). \end{aligned}$$

Now, one has

$$\begin{aligned} (B_1D - B_1G - B'_3D + B'_3G) &= (B'_1G - B_1G + B'_3G - B_3G) \\ &\quad - (B'_1D - B_1D + B'_3D - B_3D) \\ &\quad - (B_3D - B'_1D - B_3G + B'_1G) \end{aligned}$$

or

$$(B_1D - B_1G - B'_3D + B'_3G) + \omega \equiv \beta_2 - \beta_1;$$

whence

$$\begin{aligned} \Omega_i - \Omega'_i + (\zeta_1 - \zeta_3)\omega &\equiv \zeta_1(\beta_2 - \beta_1), \\ \Omega_i - \Omega'_i &\sim (\zeta_3 - \zeta_1)\omega. \end{aligned}$$

Now, we have supposed that the cycle Ω_i is invariant, that is to say, that

$$\Omega_i \sim \Omega'_i,$$

which requires $\zeta_1 = \zeta_3$. Under these conditions, let us take up again the congruence (5) of section 2, which can be written

$$\Omega_i - \Omega'_i \equiv \Sigma \theta'_k M F'_k.$$

³⁰

Bringing it together with the preceding congruence, we find

$$\Sigma \theta'_k M F'_k = \zeta_1(\beta_2 - \beta_1).$$

When y (or M) comes to A_i , the partial polygons β_1 and β_2 disappear, so that the second member of this equality disappears; the first reduces to $\Sigma \theta'_k \alpha_i F'_k$, whence one derives

$$\Sigma \theta'_k \alpha_i F'_k = 0.$$

Thus what we said on pages 181 and following is justified.

Let us now suppose that A_i is a singular point of the second sort, and that for $y = A_i$ the singular point B_1 comes to O .

We can then make a figure analogous to figure 1; to simplify, I shall represent only three singular points, B_1, B_2 and B_3 .

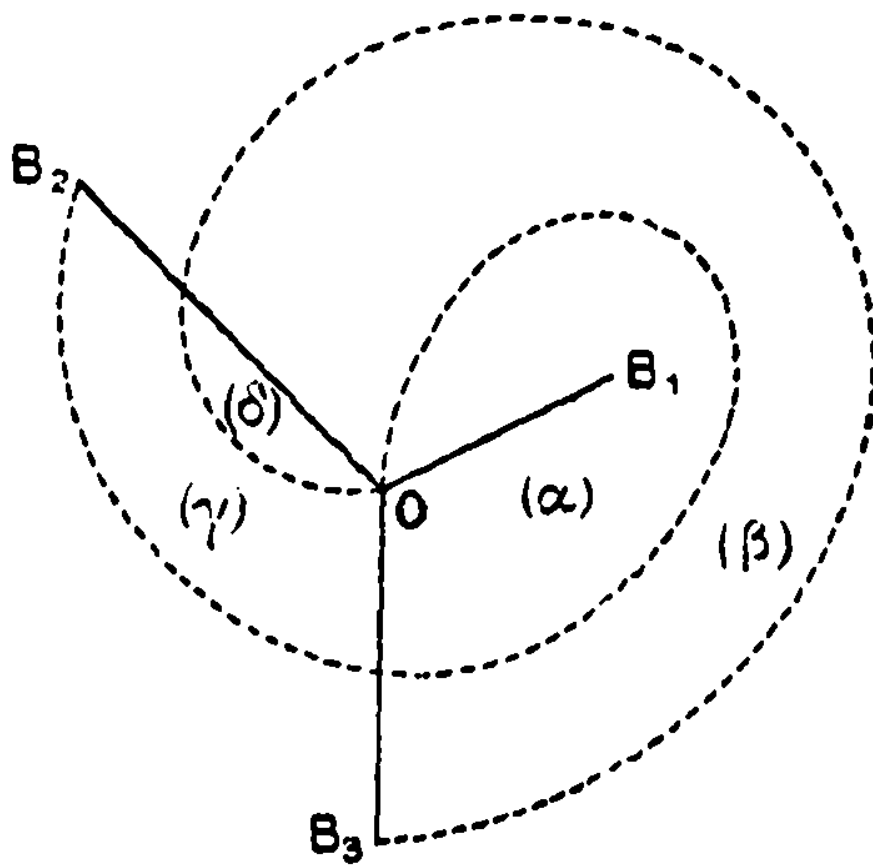


Figure 2: Fig. 2.

When y turns round A_i , the cut OB_1 resumes its original situation, but the cuts OB_2 and OB_3 (drawn solid) are transformed into the cuts OB_2 and OB_3 (drawn dotted).

We have two modes of subdivision and, by superposing them, one has the polyhedron P' .

Each face of P (that is to say, each sheet of S) is found subdivided into a certain number of partial faces. In the case of figure 2, there are five of them, designated in the figure by (α) , (β) , (γ) , (δ) , (ε) .

It is clear that, when y describes the cut OA_i , one can arrange for the preceding figure (and consequently the polyhedron P') to remain constantly homeomorphic to itself.

I shall denote by OB_2 and OB_3 the cuts drawn solid, and by OB'_2 and OB'_3 the corresponding cuts drawn dotted. I see that, these cuts cutting one another, each of the segments of these cuts will form one of the edges of P' . I shall distinguish on each of these cuts the *terminal segment*, that is to say, the one which ends at the point B_2 or B_3 .

When y comes to A_i , the polyhedron P' degenerates; several of its edges reduce to zero, namely the edge OB_1 and all the segments of OB_2 , OB_3 , OB'_2 , OB'_3 , *the terminal segments excepted*. Moreover, the terminal segment of OB_2 coincides with that of OB'_2 , and that of OB_3 with that of OB'_3 . It follows that the four partial faces (α) , (β) , (γ) , (δ) disappear.

There is no vanishing cycle here, the singular point A_i not being essential.

We have in the figure

$$\begin{aligned} OB_2 - OB'_2 &\equiv (\alpha) + (\gamma) + (\delta), \\ OB_3 - OB'_3 &\equiv (\alpha) + (\beta) + (\delta). \end{aligned}$$

Let us return to our cycle Ω_i ; we shall have

$$\Omega_i = \Sigma\zeta_1 OB_1 + \Sigma\zeta_2 OB_2 + \Sigma\zeta_3 OB_3.$$

I put the sign Σ because there are several cuts (belonging to different sheets of the Riemann surface) which project along OB_1 . One will then have

$$\Omega'_i = \Sigma\zeta_1 OB_1 + \Sigma\zeta_2 OB'_2 + \Sigma\zeta_3 OB'_3;$$

whence

$$\begin{aligned} \Omega_i - \Omega'_i &= \Sigma\zeta_2 (OB_2 - OB'_2) + \Sigma\zeta_3 (OB_3 - OB'_3), \\ \Omega_i - \Omega'_i &\equiv \Sigma\zeta_2 [(\alpha) + (\gamma) + (\delta)] + \Sigma\zeta_3 [(\alpha) + (\beta) + (\delta)], \end{aligned}$$

and, comparing with the congruence (5) of section 2,

$$\Sigma\theta'_k MF'_k = \Sigma\zeta_2 [(\alpha) + (\gamma) + (\delta)] + \Sigma\zeta_3 [(\alpha) + (\beta) + (\delta)].$$

²⁹ *Editorial note:* The last edge is printed B'_3G ; B'_1G is expected, the image of B_3G as in the permutation stated on p. 210 and in the next display; an apparent misprint.

³⁰ *Editorial note:* Congruence (5) of § 2 reads $\Sigma\theta'_k MF'_k \equiv \Omega'_i - \Omega_i$, with the opposite sign; the print does not show which form is intended.

When y comes to A_i , (α) , (γ) , (δ) disappear, so that there remains

$$\Sigma \theta'_k \alpha_i F'_k = 0,$$

which justifies here too what we said on page 181.

Let us suppose that the point y describes the different cuts $OA_1, OA_2, \dots, OA_q$; the singular points B will describe certain lines; we can suppose that none of these lines recedes indefinitely. Indeed, if this were not so, we should always find in the x -plane a small circle which would be cut by none of these lines, and then, by a simple linear change of variables, we could throw the centre of this circle to infinity.

Therefore, when the point y describes successively the two lips of OA_1 , the two lips of OA_2 , etc., and finally those of OA_q , the singular points B will remain at a finite distance. We can therefore direct the deformation of the cuts OB in such a way that these cuts remain at a finite distance. The cycle Ω_i , which is formed by a combination of these cuts, will therefore always remain at a finite distance; and the same will hold for the cycle U_i which corresponds to it on S_0 [for we can choose the correspondence of $S(M)$ with S_0 in such a way that to a point at infinity on $S(M)$ there corresponds a point at infinity on S_0]. This cycle cannot, therefore, sweep out the entire surface S_0 , which justifies what we said on page 185.

It will be remarked, finally, that among the vertices of P we have m which correspond to $x = 0$.

All our hypotheses are therefore justified.