Jac. B. Solution of Leibniz's Problem

on the Curve of uniform Approach to and Recession from a given point, by means of the rectification of the Elastic Curve

The felicity of the preceding discovery may be commended by the solution of that most elegant *Leibnizian* Problem, of finding the *line along which a descending heavy body recedes uniformly, in equal times, from a given point, or approaches it*: which so pleased its most praised Author that he not only several times challenged Friends and Adversaries alike, in the *Acta*, to attempt it (see *Anno* 1689, *Apr. p.* 198, and 1690. *May p.* 229, and *Jul. p.* 360,) but himself also strenuously toiled at it, as witnessed by several properties of the Curve which he had had inserted in the *French Journal*, as I have from the report of my *Brother*, who nevertheless could not name them to me. My *Brother* himself, too, was much occupied with the same thing while he was staying at *Paris*, but with all his inverse Method of Tangents he accomplished nothing else than to reduce the Problem to this differential equation: $(x\,dx + y\,dy)\sqrt{y} = (x\,dy - y\,dx)\sqrt{a}$, to which, however, one arrives even with no labour. But its full solution, or construction, neither he nor anyone else was able to give. Recently, while I was preparing the preceding essay for *Leipzig*, and had already committed to paper part of the discovery together with the figures themselves, I undertook the scrutiny of this Line, on the occasion of the new Theorems which I gave there for the Radii of the osculating circles in Spirals (whose investigation, however, was itself made only incidentally; for the chief points of the discovery were only coming to birth under my pen) soon after a few attempts I attained the wished-for end; whereupon this most singular thing befell me, that, the analysis being fully worked out, I noticed that the Elastic curve, which had given only some chance occasion to this attempt, was that very one by whose rectification the other could be constructed. For although the same might be accomplished by means of the quadrature of some algebraic space, I nevertheless judge the other mode of construction to be preferred, both because in general curves are more easily rectified in practice than spaces are squared, and especially because nature herself (provided she observes a suitable law of tension) seems to have prescribed that one.

Constr. The Elastic curve of the third construction (*Fig.* 3) having therefore been described, and the rest being placed as before, let there be applied, in each quadrant of the circle BL, LG, the straight line $\varepsilon\zeta$ equal to Ag itself, that is, to the third proportional to AB and AE (here it appears drawn only in the right quadrant, so that confusion of the lines may be avoided) and from the drawn $A\varepsilon$ (produced, if there be need) let there be cut off $A\alpha$ equal to the third proportional to the straight line AB and the

portion of the curve AQ in the left, or φRQ in the right quadrant; and the point α will be on a certain Curve $A\chi\omega\eta$, so contrived that a heavy body carried along it (if it has first fallen from the height iA) approaches the point A uniformly in equal times, or recedes from it. The demonstration of this construction (so that I may relieve the most Celebrated Author of the Problem, most busied with other matters, of the labour of examining it) I here append.

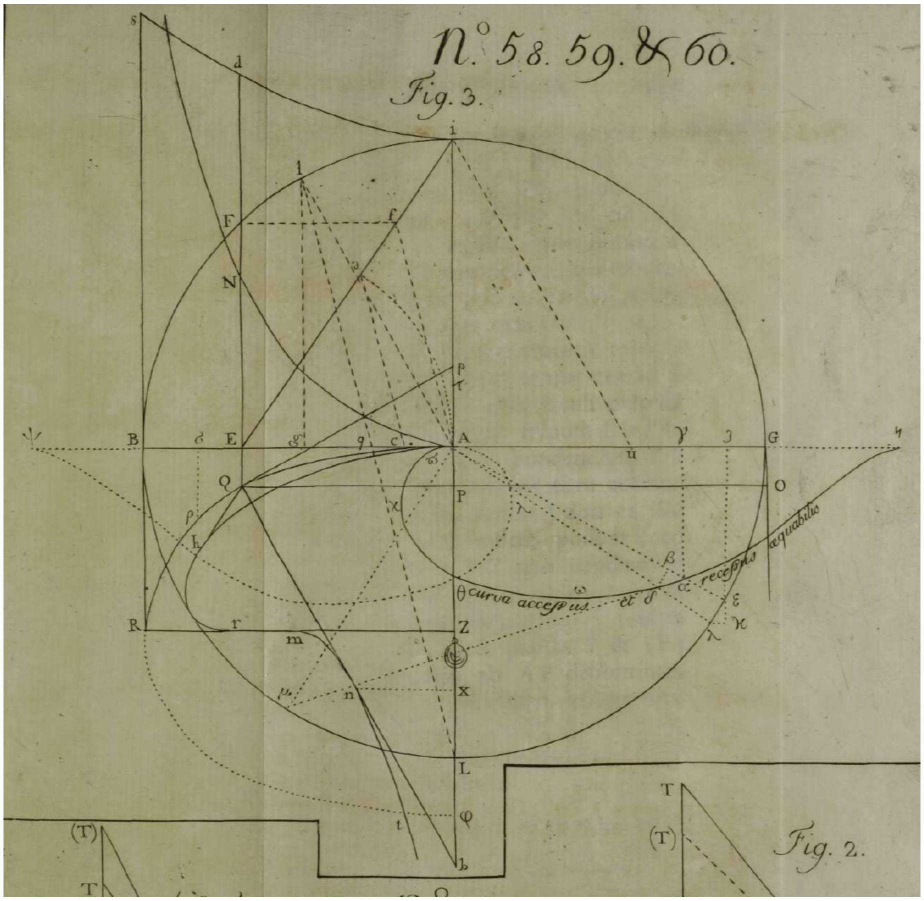

Figure 1: Fig. 3.

Dem. Let a little circle $\beta\delta$ be understood as described with centre A, cutting off from the applied line $A\alpha$ the Element $\alpha\beta$, and from the Curve the Element $\alpha\delta$, at whose vertex let the isochronous little portion $A\omega$ be conceived. Now the Ratio of $\beta\delta$ to $\alpha\beta$ is compounded of the following 5 ratios:

$\beta\delta$: Elem. periph. $B\varepsilon$ :: $A\alpha : A\varepsilon$.

Elem. periph. $B\varepsilon$: Elem. $\varepsilon\zeta$ :: $A\varepsilon : A\zeta$, ex natura Circuli.

Elem. $\varepsilon\zeta$: Elem. AE :: $2AE : AB$, per constr. & princ. calcul. in fin. parvorum.

Elem. AE : Elem. AQ (φRQ) :: $A\zeta : AB$, ex nat. curvae AQ, ut collig. ex coroll. 2. constructionis tertia

Elem. AQ (φRQ) : $\alpha\beta$ :: $AB : 2AQ$ ($2\varphi RQ$), per constr. & princ. calc. inf. parv.

whence, the division being made by the common antecedents and consequents of the ratios, $\beta\delta : \alpha\beta :: A\alpha : AB + AE : AQ$ (φRQ,) and $\beta\delta^2 : \alpha\beta^2 :: A\alpha^2 : AB^2 + AE^2 : AQ^2$ or φRQ^2 ($\varepsilon\zeta : A\alpha$, by constr.) $:: A\alpha^2 \times \varepsilon\zeta : AB^2 \times A\alpha :: A\alpha \times \varepsilon\zeta : AB^2 ::$ (on account of the similarity of the Triangles $A\alpha\gamma$, $A\varepsilon\zeta$) $A\varepsilon \times \alpha\gamma : AB^2 :: \alpha\gamma : AB$ (Ai), and, by composition, $\alpha\delta^2 : \alpha\beta^2 :: \alpha\gamma + Ai : Ai$; and accordingly $\alpha\delta : \alpha\beta :: \sqrt{\alpha\gamma + Ai} : \sqrt{Ai} ::$ the velocity acquired at α : the velocity acquired at A :: (from the nature of the descent of heavy bodies) $:: \alpha\delta : A\omega$ (by hypothesis, since they are supposed to be traversed in equal little times). Therefore $\alpha\beta = A\omega$. Therefore a heavy body carried along this Curve recedes uniformly, in equal times, from the point A. Q. E. D.

The things chiefly to be observed here are the following:

1. If the heavy body be supposed to fall from higher or lower than from i, say from τ, there is no need of a new Elastic curve: this one alone suffices for describing all the isochronous Curves. For they are all similar to one another, and are determined merely by making, as Ai to $A\tau$, so $A\alpha$ to another $A\alpha$ to be cut off from the same $A\varepsilon$. And the curve of our Figure is itself constructed for a heavy body falling as though from τ, not from i.

2. But there are also infinitely many others which a heavy body, having slipped down from the same height

τA, can traverse so as to approach or recede uniformly from some point; but that point continually different from A; nor, except by a single curve from the same height, with respect to the same point, can uniform approach or recession be brought about.

3. Our Isochronous Curve takes its beginning at the very point A, with respect to which the motion is uniform, and is terminated at the point η of the same height as it; each extremity indeed touches the horizontal straight line $A\eta$. Whence it is established to have a contrary flexure. But if on the other side there be set up a curve $A\lambda\theta\psi$ similar and equal to this, and the heavy body be released from η with the velocity which it can acquire by falling from a height equal to τA itself, then, first by descending to ω, then by re-ascending, it will approach A uniformly, and thence, going on continually through the other $A\lambda\theta\psi$, it will be drawn away from the same point A by the same law.

4. The tangent of the Curve at intermediate places, such as α, is found if, having drawn $A\mu$ perpendicular to $A\alpha$, one makes, as $\sqrt{A\tau}$ to $\sqrt{\alpha\gamma}$, so $A\alpha$ to $A\mu$; for $\alpha\mu$ being drawn will touch the curve: which is easily inferred even from the use which the curve ought to serve.

5. The straight line $A\eta$ is quadruple the straight line $A\theta$.

6. If on the Elastic curve a point Q be taken such that the third proportional to the subtangent Pp and the straight line AB equals the curve AQ, there will be found by its means the point χ of the portion $A\chi\theta$ which of all is the most distant from the perpendicular $A\theta$.

7. But if the point Q be placed such that the tangent Qp equals the curve φRQ, by its means there will be had, on the other curve, the point ω, the lowest of all, and the most remote from the horizontal $A\eta$.

8. Finally, this too is to be noted, that from the construction of the proposed Problem it is now easy to gather what suppositions are to be made in order to reduce the equation $(x\,dx + y\,dy)\sqrt{y} = (x\,dy - y\,dx)\sqrt{a}$. For if in place of x and y there be assumed two other indeterminates t and z, by putting $ay = tz$, and $ax = t\sqrt{aa - zz}$; there will come forth an equation in which the indeterminates t and z, with their differentials, can be separated from one another; which suffices for then expediting the construction, at least by quadratures. For two things in this calculus seem still chiefly to be wanting, which, if they could always be done, everything would be found out; one, that differentials of the second or higher kinds be reduced to differentials of the first; the other, that in differential equations of the first

kind the indeterminates, if they be mixed together, be mutually separated from one another, so that each with its own differential may constitute a particular part of the equation; that is, that the inverse method of tangents be found. For each I have given certain Rules (and I could give more without end), similar to those which my *Brother* deposited at *Paris* with the *Marquis* de l'Hospital, in which he had at first supposed a method to reside. But I at once perceived that they contained nothing but certain particular artifices, which I would not dare to call a method; inasmuch as I judge that no such thing can be given, any more than a universal method can be given for constructing algebraic Equations of any degrees whatsoever indiscriminately. For that one (of which mention was once made in the *Acta*), by which I posit indefinitely $0 = a + bx + cy + exy$, etc. and then seek by its means the tangent of the line, seems scarcely to succeed otherwise than in speculation. That this may lend credence to what has been said, let it be this: that the present Problem, although as it appears not very difficult, could be solved by no one by the aid of any rule or method hitherto discovered.